Proof of "Continuous Group-wise Double Auction for Prosumers in Distribution-level Markets"

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Appendix A
Proof in Section III

Proof of Proposition 1 (Truthfulness). Without loss of generality, we assume that prosumer i is a buyer. Suppose that the prosumer receives a clearing price c_i^* with the reference power $\bar{\tau}_k$ when submitting a truthful bidding curve $p_0^{i*}(c)$. Likewise, suppose that the clearing price and reference power are $c_i^{\tau*}$ and $\tilde{\tau}_k$, respectively, if the prosumer submits an untruthful bidding. Suppose that $\tilde{c}_i^\tau$ is the price at which the optimal trading power of prosumer i is equal to $\tau_i(c_i^{\tau*})$. That is, $p_0^{i*}(\tilde{c}_i^\tau)=\tau_i(c_i^{\tau*})$.

The first scenario concerns an untruthful bidding that makes the prosumer change its role from a buyer to a seller. Then, we can show that the prosumer's payoff becomes non-positive as follows.

$$
\begin{aligned}
&U_i(c_i^{\tau*},\{\tau_i(c_i^{\tau*})\}^-,\beta_i)\\
=&\int_{c_i^{\tau*}}^{\bar{c}_k}\{\tilde{\tau}_k+\sum_{i'\in S_k}\{\tau_{i'}(c)\}^-\}^+\mathrm{d}c\\
&+\int_{\tilde{c}_i^\tau}^{c_i^0}\{p_0^{i*}(c)\}^-\mathrm{d}c+(\tilde{c}_i^\tau-c_i^{\tau*})\{\tau_i(c_i^{\tau*})\}^- &\text{(1a)}\\
\le&(\bar{c}_k-c_i^{\tau*})\big\{\tilde{\tau}_k+\sum_{i'\in S_k}\{\tau_{i'}(c_i^{\tau*})\}^-\big\}^+\\
&+(c_i^0-\tilde{c}_i^\tau)\{p_0^{i*}(\tilde{c}_i^\tau)\}^-+(\tilde{c}_i^\tau-c_i^{\tau*})\{\tau_i(c_i^{\tau*})\}^- &\text{(1b)}\\
=&(c_i^0-\bar{c}_k)\{\tau_i(c_i^{\tau*})\}^- &\text{(1c)}\\
\le&0, &\text{(1d)}
\end{aligned}
$$

where (1b) is due to the non-increasing properties of $\{\tilde{\tau}_k+\sum_{i'\in S_k}\{\tau_{i'}(c)\}^-\}^+$ and $p_0^{i*}(c)$. Hence, the prosumer does not have the incentive to change its role from a buyer to a seller by untruthful bidding. Likewise, we can prove that a prosumer has no incentive to change its role from a seller to a buyer either.

In the second scenario, we prove that a prosumer achieves less payoff by submitting an untruthful bidding, even if the untruthful bidding does not change its role.

Suppose that prosumer i is a buyer and its truthful bidding makes its reference power depend on the buyers. That is, $\sum_{i'\in B_k\setminus i}\tau_{i'}(\bar{c}_k)+p_0^{i*}(\bar{c}_k)\le\sum_{i'\in S_k}-\tau_{i'}(\bar{c}_k)$. If an untruthful bidding increases the reference power, i.e., $\sum_{i'\in B_k\setminus i}\tau_{i'}(\bar{c}_k)+\tau_i(\bar{c}_k)>\sum_{i'\in S_k}-\tau_{i'}(\bar{c}_k)$, the prosumer will achieve a smaller payoff, as follows.

$$
\begin{aligned}
&U_i(c_i^{\tau*},\{\tau_i(c_i^{\tau*})\}^+,\beta_i)-U_i(c_i^*,\{p_0^{i*}(c_i^*)\}^+,\beta_i)\\
=&\int_{\bar{c}_k}^{c_i^{\tau*}}\big\{\tilde{\tau}_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(c)\}^+\big\}^+\mathrm{d}c+\int_{\tilde{c}_i^\tau}^{c_i^0}\{p_0^{i*}(c)\}^+\mathrm{d}c\\
&-\int_{\bar{c}_k}^{c_i^0}\{p_0^{i*}(c)\}^+\mathrm{d}c+(\tilde{c}_i^\tau-c_i^{\tau*})\{\tau_i(c_i^{\tau*})\}^+ &\text{(2a)}\\
\le&(c_i^{\tau*}-\bar{c}_k)\big\{\tilde{\tau}_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(c_i^{\tau*})\}^+\big\}^+\\
&+(\bar{c}_k-\tilde{c}_i^\tau)\{p_0^{i*}(\tilde{c}_i^\tau)\}^--(c_i^{\tau*}-\tilde{c}_i^\tau)\{\tau_i(c_i^{\tau*})\}^+ &\text{(2b)}\\
=&(c_i^{\tau*}-\bar{c}_k)\{\tau_i(c_i^{\tau*})\}^++(\bar{c}_k-c_i^{\tau*})\{\tau_i(c_i^{\tau*})\}^+ &\text{(2c)}\\
=&0, &\text{(2d)}
\end{aligned}
$$

where equation (2b) is based on the non-decreasing property of $\{\tilde{\tau}_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(c)\}^+\}^+$ and non-increasing property of $p_0^{i*}(c)$. Likewise, when an untruthful bidding reduces the reference power, the prosumer will achieve a smaller payoff as the operating cost increases and the reward decreases. The same conclusion also holds when the prosumer is a seller and the reference power depends on the sellers with truthful bidding.

Suppose that prosumer i is a buyer and its truthful bidding makes its reference power depend on the sellers. That is, $\sum_{i'\in B_k\setminus i}\tau_{i'}(\bar{c}_k)+p_0^{i*}(\bar{c}_k)\ge\sum_{i'\in S_k}-\tau_{i'}(\bar{c}_k)$. If an untruthful bidding decreases the reference power, i.e., $\sum_{i'\in B_k\setminus i}\tau_{i'}(\bar{c}_k)+\tau_i(\bar{c}_k)<\sum_{i'\in S_k}-\tau_{i'}(\bar{c}_k)$, the prosumer will achieve a smaller payoff, as follows.

$$
\begin{aligned}
&U_i(c_i^{\tau*},\{\tau_i(c_i^{\tau*})\}^+,\beta_i)-U_i(c_i^*,\{p_0^{i*}(c_i^*)\}^+,\beta_i)\\
=&\int_{\tilde{c}_i^\tau}^{c_i^0}\{p_0^{i*}(c)\}^+\mathrm{d}c-\int_{\bar{c}_k}^{c_i^*}\big\{\tau_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(c)\}^+\big\}^+\mathrm{d}c\\
&+(\tilde{c}_i^\tau-\bar{c}_k)\{\tau_i(\bar{c}_k)\}^+-\int_{c_i^*}^{c_i^0}\{p_0^{i*}(c)\}^+\mathrm{d}c &\text{(3a)}\\
\le&(c_i^*-\tilde{c}_i^\tau)\{p_0^{i*}(\tilde{c}_i^\tau)\}^+\mathrm{d}c_i+(\tilde{c}_i^\tau-\bar{c}_k)\{\tau_i(\bar{c}_k)\}^+\\
&-(c_i^*-\bar{c}_k)\big\{\tau_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(\bar{c}_k)\}^+\big\}^+ &\text{(3b)}\\
\le&0, &\text{(3c)}
\end{aligned}
$$

where equation (3b) is based on the non-decreasing property of $\{\tau_k-\sum_{i'\in B_k\setminus i}\{\tau_{i'}(c)\}^+\}^+$ and non-increasing property of $p_0^{i*}(c)$. Likewise, when an untruthful bidding does not change the reference power, the prosumer will achieve a smaller payoff as the operating cost increases and the reward decreases. The same conclusion also holds when the prosumer is a seller and the reference power depends on the buyers with truthful bidding.

Hence, truthful bidding is a dominant strategy for each prosumer regardless of how other prosumers bid.

□

APPENDIX B
PROOF IN SECTION III

Proof of Proposition 4 (Asymptotic Efficiency). In the distribution market, I prosumers are involved in the auction. In the centralized control, the DSO maximize the total payoff of all prosumers and the DSO by optimizing the clearing decisions (assuming the truthfulness of the prosumers).

$$\begin{aligned} \max_{c_i,\tau_i^M,\beta_i} & \sum_i U_i(c_i,\tau_i^M,\beta_i) + U_{DSO}(c_i,\tau_i^M,\beta_i) \\ \text{s.t.} & \sum_i \tau_i^M = 0. \end{aligned} \tag{4}$$

After simplification, we have

$$\begin{aligned} \max_{c_i} & \sum_i [f_i^*(c_i^0) - f_i^*(c_i^*)] \\ \text{s.t.} & \sum_i p_0^{i*}(c_i) = 0. \end{aligned} \tag{5}$$

Recall the definition of $f_i^*(c_i)$

$$\begin{aligned} f_i^*(c_i) = \min_{\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i} & f_i(\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i;c_i^*) \\ \text{s.t.} \quad & \{\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i\} \in \mathcal{X}^i. \end{aligned} \tag{6}$$

The total payoff achieve in (5) is the same as

$$\begin{aligned} \max_{c,\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i} & \sum_i [f_i^*(c_i^0) - f_i(\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i;c)] \\ \text{s.t.} & \sum_i p_0^{i*} = 0, \\ & \{\boldsymbol{x}^i,p_0^i,g_j^i,\Delta d_j^i\} \in \mathcal{X}^i. \end{aligned} \tag{7}$$

According to the Lemma 3 in [1], if all prosumers optimize their operating cost according to the market clearing price c^* that satisfies $\sum_i p_0^{i*}(c^*) = 0$, the achieved power dispatch is the same as the optimal one achieved by centralized control. Let c^* be optimal market clearing price such that $\sum_i \tau_i(c^*) = 0$. Then, the trade outcome in the centralized optimization scheme is computed as $\sum_i [f_i^*(c_i^0) - f_i^*(c^*)]$.

In order to prove Lemma. 4, we need to show $\lim_{I\to\infty} \frac{\sum_i U_i(c_i^*,\tau_i^*,\beta_i)+U_{DSO}}{\sum_i\{f_i^*(c_i^0)-f_i^*(c^*)\}} = 1$, which is equivalent to prove $\lim_{I\to\infty} \frac{\sum_i\{f_i^*(c_i^0)-f_i^*(c^*)\}-\sum_i U_i(c_i^*,\tau_i^*,\beta_i)-U_{DSO}}{\sum_i\{f_i^*(c_i^0-f_i^*(c^*))\}} = 0$. If the prosumers optimal price responses are independently and identically distributed, then the following two Lemmas hold.

Lemma B.1. *The trade outcome in the centralized optimization scheme is $\mathcal{O}(I)$ for a large I.*

Proof. Since all optimal price responses are independently and identically distributed, the expectation of the total payoffs of $2n$ prosumers is no less than that of n individual groups with each contains two prosumers. Moreover, the expectation of the total payoffs of any two prosumers are the same. Hence, we can conclude that the trade outcome increases at least linearly with the size of prosumers, i.e., the trade outcome is $\mathcal{O}(I)$ for a large I. □

Lemma B.2. *The bidding curves $p_0^{i*}(c_i)$ are independent under the same price, and bounded (with boundary $[-\theta,\theta]$).*

Lemma B.2 follows directly from the assumption.

As all optimal price responses are independent, not all prosumers have the horizontal price responses at the same price interval. If at least $I^\gamma, \gamma \in (0,1]$ prosumers have optimal price responses that have non-zero slops, then we can have the following assumption.

Assumption 0.1. *As $I \to \infty$, the slope of I cumulated bidding curves is bounded by $[aI^\gamma, bI]$, where $a,b<0$ and $\gamma \in (0,1]$.*

In the proposed mechanism, the prosumers are divided into $K = I^\alpha, \alpha \in (0,1)$ groups, i.e., each group contains $I^{1-\alpha}$ prosumers. We consider that $\{p_0^{i*}(c^*)\}_{i\in\mathcal{I}}$ construct a probability distribution with mean to be 0, i.e., $p_0^{i*}(c^*) \sim (0,\sigma^2)$. As $p_0^{i*}(c^*)$ is bounded in $[-\theta,\theta]$, $\sigma^2 \le \theta^2$. According to Lemma B.2, we have

$$\sum_{i\in G_k} p_0^{i*}(c^*) \sim (0, I^{1-\alpha}\sigma^2). \tag{8}$$

According to Chebyshev's inequality,

$$\Pr(|\sum_{i\in G_k} p_0^{i*}(c^*)| \ge \delta\sqrt{I^{1-\alpha}\sigma^2}) \le \frac{1}{\delta^2}, \forall\delta. \tag{9}$$

When $\delta = I^\lambda, \lambda \in (0,1)$, $\Pr(|\sum_{i\in G_k} p_0^{i*}(c^*)| \ge I^{\frac{1-\alpha}{2}+\lambda}\theta) \le I^{-2\lambda}$. As $\sum_{i\notin G_k} p_0^{i*}(c^*) + \sum_{i\in G_k} p_0^{i*}(c^*) = 0$,

$$\Pr(|\sum_{i\notin G_k} p_0^{i*}(c^*)| \ge I^{\frac{1-\alpha}{2}+\lambda}\theta) \le I^{-2\lambda}. \tag{10}$$

Hence, with more than $1 - I^{-2\lambda}$ probability, $|\sum_{i\notin G_k} p_0^{i*}(c^*)| \le I^{\frac{1-\alpha}{2}+\lambda}\theta$. Then, according to Assumption 0.1, the difference between optimal market price c^* and the reference price for group G_k is

$$|c^* - \bar{c}_k| \le \frac{I^{\frac{1-\alpha}{2}+\lambda}\theta}{aI^\gamma} = \Theta(I^{\frac{1-\alpha}{2}+\lambda-\gamma}) \tag{11}$$

with same probability.

Without loss of generalization, we assume that the reference trading power depends on the buyers in group G_k, i.e.,

$\sum_{i\in B_k} p_0^{i*}(\bar{c}_k) = \bar{\tau}_k$. The difference of the trade outcome between both mechanisms is

$$\begin{aligned}
&\sum_i [f_i^*(c_i^0) - f_i^*(c^*)] - \sum_i U_i(c_i^*, \tau_i^*, \beta_i) - U_{DSO}(c_i, \tau_i^M, \beta_i) \\
=&\sum_k \Big\{ \sum_{i\in G_k} \int_{c^*}^{c_i^0} p_0^{i*}(c)\mathrm{d}c - \sum_{i\in B_k} \int_{c_k^b}^{c_i^0} \{p_0^{i*}(c)\}^+ \mathrm{d}c \\
&\quad - \sum_{i\in S_k} \int_{c_k^s}^{c_i^0} \{p_0^{i*}(c)\}^- \mathrm{d}c - (c_k^b - c_k^s)\bar{\tau}_k \Big\} \\
=&\sum_k \Big\{ \sum_{i\in B_k} \int_{c^*}^{c_k^b} \{p_0^{i*}(c)\}^+ \mathrm{d}c + \sum_{i\in S_k} \int_{c^*}^{c_k^s} \{p_0^{i*}(c)\}^- \mathrm{d}c - (c_k^b - c_k^s)\bar{\tau}_k \\
&+ \sum_{i\in B_k} \int_{c^*}^{c_i^0} \{p_0^{i*}(c)\}^- \mathrm{d}c + \sum_{i\in S_k} \int_{c_k^b}^{c_i^0} \{p_0^{i*}(c)\}^+ \mathrm{d}c \Big\} \\
\le&\sum_k \Big\{ (c_k^b - c^*)[\sum_{i\in B_k} p_0^{i*}(c^*) - \bar{\tau}_k] + (c^* - c_k^s)[-\sum_{i\in S_k} p_0^{i*}(c^*) - \bar{\tau}_k] \Big\}.
\end{aligned} \tag{12}$$

Then, according to Assumption 0.1, the difference of the trade outcome is bounded up by

$$\begin{aligned}
&I^\alpha \frac{(\sum_{i\in G_k} p_0^{i*}(c^*) + \frac{\sum_{i\notin G_k} p_0^{i*}(c^*)}{aI^\gamma} bI^{(1-\alpha)})^2}{aI^{(1-\alpha)\gamma}} \\
=&\begin{cases} \Theta(I^{1+2\lambda-(1-\alpha)\gamma}), & \text{if } 1-\alpha < \gamma \\ \Theta(I^{3+2\lambda-2\alpha-(3-\alpha)\gamma}), & \text{if } 1-\alpha \ge \gamma \end{cases}.
\end{aligned} \tag{13}$$

θ can have value at most 0. When $1-\alpha < \gamma$, we have

$$\begin{aligned}
&\lim_{I\to\infty} \frac{\sum_i \{f_i^*(c_i^0) - f_i^*(c^*) - U_i(c_i^*, \tau_i^*, \beta_i)\} - U_{DSO}}{\sum_i \{f_i^*(c_i^0) - f_i^*(c^*)\}} \\
=&\lim_{I\to\infty} \Omega(I^{-(1-\alpha)\gamma}) = 0.
\end{aligned} \tag{14}$$

When $1-\alpha \ge \gamma$, we have

$$\begin{aligned}
&\lim_{I\to\infty} \frac{\sum_i \{f_i^*(c_i^0) - f_i^*(c^*) - U_i(c_i^*, \tau_i^*, \beta_i)\} - U_{DSO}}{\sum_i \{f_i^*(c_i^0) - f_i^*(c^*)\}} \\
=&\lim_{I\to\infty} \Omega(I^{2-2\alpha-(3-\alpha)\gamma}).
\end{aligned} \tag{15}$$

In order to guarantee $2-2\alpha-(3-\alpha)\gamma < 0$, $1-\gamma > \alpha > \frac{2-3\gamma}{2-\gamma}$. Hence, when $\alpha > \frac{2-3\gamma}{2-\gamma}$, the trade outcome of the proposed CGDA mechanism converges to the maximum one achieved by centralized optimization as the number of prosumers goes to infinity. □

REFERENCES

[1] A. Yu, C. Zhang, and Y. A. Zhang, "Optimal bidding strategy of prosumers in distribution-level energy markets," *IEEE Transactions on Power Systems*, vol. 35, no. 3, pp. 1695–1706, 2020.